

One-Dimensional Peg Solitaire

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Abstract. We solve the problem of one-dimensional peg solitaire. In particular, we show that the set of configurations that can be reduced to a single peg forms a regular language, and that a linear-time algorithm exists for reducing any configuration to the minimum number of pegs.

1 Introduction

Peg solitaire is a game for one player. Each move consists of hopping a peg over another one, which is removed. The goal is to reduce the board to a single peg. The best-known forms of the game take place on cross-shaped or triangular boards, and it has been marketed as “Puzzle Pegs” and “Hi-Q.” Discussions and various solutions can be found in [1,2,3,4,5].

In [6], Guy proposes one-dimensional peg solitaire as an open problem in the field of combinatorial games. Here we show that the set of solvable configurations forms a regular language, i.e. it can be recognized by a finite-state automaton. In fact, this was already shown in 1991 by Plambeck ([7], Introduction and Ch.5) and appeared as an exercise in a 1974 book of Manna [8]. More generally, B. Ravikumar showed that the set of solvable configurations on rectangular boards of any finite width is regular [9], although finding an explicit grammar seems to be difficult on boards of width greater than 2.

Thus there is little new about this result. However, it seems not to have appeared in print, so here it is.

Theorem 1. *The set of configurations that can be reduced to a single peg is the regular language 0^*L0^* where*

$$\begin{aligned} L = & 1 + 011 + 110 \\ & + 11(01)^* \left[00 + 00(11)^+ + (11)^+00 + (11)^*1011 + 1101(11)^* \right] (10)^*11 \\ & + 11(01)^*(11)^*01 + 10(11)^*(10)^*11. \end{aligned} \tag{1}$$

Here 1 and 0 indicate a peg and a hole respectively, w^* means ‘0 or more repetitions of w ’, and $w^+ = ww^*$ means ‘1 or more repetitions of w .’

Proof. To prove the theorem, we follow Leibnitz [4] in starting with a single peg, which we denote

$$1$$

and playing the game in reverse. The first ‘unhop’ produces

$$011 \text{ or } 110$$

and the next

$$1101 \text{ or } 1011.$$

(As it turns out, 11 is the only configuration that cannot be reduced to a single peg without using a hole outside the initial set of pegs. Therefore, for all larger configurations we can ignore the 0’s on each end.)

We take the second of these as our example. It has two ends, $10\dots$ and $\dots 11$. The latter can propagate itself indefinitely by unhopping to the right,

$$1010101011.$$

When the former unhops, two things happen; it becomes an end of the form $11\dots$ and it leaves behind a space of two adjacent holes,

$$110010101011.$$

Furthermore, this is the only way to create a 00. We can move the 00 to the right by unhopping pegs into it,

$$111111110011.$$

However, since this leaves a solid block of 1’s to its left, we cannot move the 00 back to the left. Any attempt to do so reduces it to a single hole,

$$1111111101111.$$

Here we are using the fact that if a peg has another peg to its left, it can never unhop to its left. We prove this by induction: assume it is true for pairs of pegs farther left in the configuration. Since adding a peg never helps another peg unhop, we can assume that the two pegs have nothing but holes to their left. Unhopping the leftmost peg then produces 1101, and the original (rightmost) peg is still blocked, this time by a peg which itself cannot move for the same reason.

In fact, there can never be more than one 00, and there is no need to create one more than once, since after creating the first one the only way to create another end of the form $10\dots$ or $\dots 01$ is to move the 00 all the way through to the other side

$$111111111101$$

and another 00 created on the right end now might as well be the same one.

We can summarize, and say that any configuration with three or more pegs that can be reduced to a single peg can be obtained in reverse from a single peg by going through the following stages, or their mirror reflection:

1. We start with 1011. By unhopping the rightmost peg, we obtain $10(10)^*11$. If we like, we then
2. Unhop the leftmost peg one or more times, creating a pair of holes and obtaining $11(01)^*00(10)^*11$. We can then
3. Move the 00 to the right (say), obtaining $11(01)^*(11)^*00(10)^*11$. We can stop here, or
4. Move the 00 all the way to the right, obtaining $11(01)^*(11)^*01$, or
5. Fill the pair by unhopping from the left, obtaining $11(01)^*(11)^*1011(10)^*11$.

Equation 1 simply states that the set of configurations is the union of all of these plus 1, 011, and 110, with as many additional holes on either side as we like. Then 0^*L0^* is regular since it can be described by a regular expression [10], i.e. a finite expression using the operators $+$ and $*$. \square

Among other things, Theorem 1 allows us to calculate the number of distinct configurations with n pegs, which is

$$N(n) = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ 2 & n = 3 \\ 15 - 7n + n^2 & n \geq 4, n \text{ even} \\ 16 - 7n + n^2 & n \geq 5, n \text{ odd} \end{cases}$$

Here we decline to count 011 and 110 as separate configurations, since many configurations have more than one way to reduce them.

We also have the corollary

Corollary 1. *There is a linear-time strategy for playing peg solitaire in one dimension.*

Proof. Our proof of Theorem 1 is constructive in that it tells us how to unhop from a single peg to any feasible configuration. We simply reverse this series of moves to play the game. \square

More generally, a configuration that can be reduced to k pegs must belong to the regular language $(0^*L0^*)^k$, since unhopping cannot interleave the pegs coming from different origins [7]. This leads to the following algorithm:

Theorem 2. *There is a linear-time strategy for reducing any one-dimensional peg solitaire configuration to the minimum possible number of pegs.*

Proof. Suppose we are given a string $c_0c_1c_2 \dots c_{n-1}$ where each $c_i \in \{0, 1\}$. Let \mathcal{A} be a nondeterministic finite automaton (without ϵ -transitions) for 0^*L0^* , where A is the set of states in \mathcal{A} , s is the start state, and T is the set of accepting states. We then construct a directed acyclic graph G as follows: Let the vertices of G consist of all pairs (a, i) where $a \in A$ and $0 \leq i \leq n$. Draw an arc from (a, i) to $(b, i+1)$ in G whenever \mathcal{A} makes a transition from state a to state b on symbol c_i . Also, draw an arc from (t, i) to (s, i) for any $t \in T$ and any $0 \leq i \leq n$. Since $|\mathcal{A}| = \mathcal{O}(1)$, $|G| = \mathcal{O}(n)$.

Then any path from $(s, 0)$ to (s, n) in G consists of n arcs of the form (a, i) to $(b, i + 1)$, together with some number k of arcs of the form (t, i) to (s, i) . Breaking the path into subpaths by removing all but the last arc of this second type corresponds to partitioning the input string into substrings of the form 0^*L0^* , so the length of the shortest path from $(s, 0)$ to (s, n) in G is $n + k$, where k is the minimum number of pegs to which the initial configuration can be reduced. Since G is a directed acyclic graph, we can find shortest paths from $(s, 0)$ by scanning the vertices (a, i) in order by i , resolving ties among vertices with equal i by scanning vertices (t, i) (with $t \in T$) earlier than vertex (s, i) . When we scan a vertex, we compute its distance to $(s, 0)$ as one plus the minimum distance of any predecessor of the vertex. If the vertex is $(s, 0)$ itself, the distance is zero, and all other vertices $(a, 0)$ have no predecessors and infinite distance.

Thus we can find the optimal strategy for the initial peg solitaire configuration by forming G , computing its shortest path, using the location of the edges from (t, i) to (s, i) to partition the configuration into one-peg subconfigurations, and applying Corollary 1 to each subconfiguration. Since $|G| = \mathcal{O}(n)$, this algorithm runs in linear time.

In contrast to these results, Uehara and Iwata [11] showed that in two or more dimensions peg solitaire is NP-complete. However, the complexity of finding the minimum number of pegs to which a $k \times n$ configuration can be reduced, for bounded $k > 2$, remains open.

Finally, we note that Ravikumar has proposed an impartial two-player game, in which players take turns making peg solitaire moves, and whoever is left without a move loses. It is tempting to think that this two-player game might be PSPACE-complete in two or more dimensions, and polynomial-time solvable in one.

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